

Surfaces and Calculus

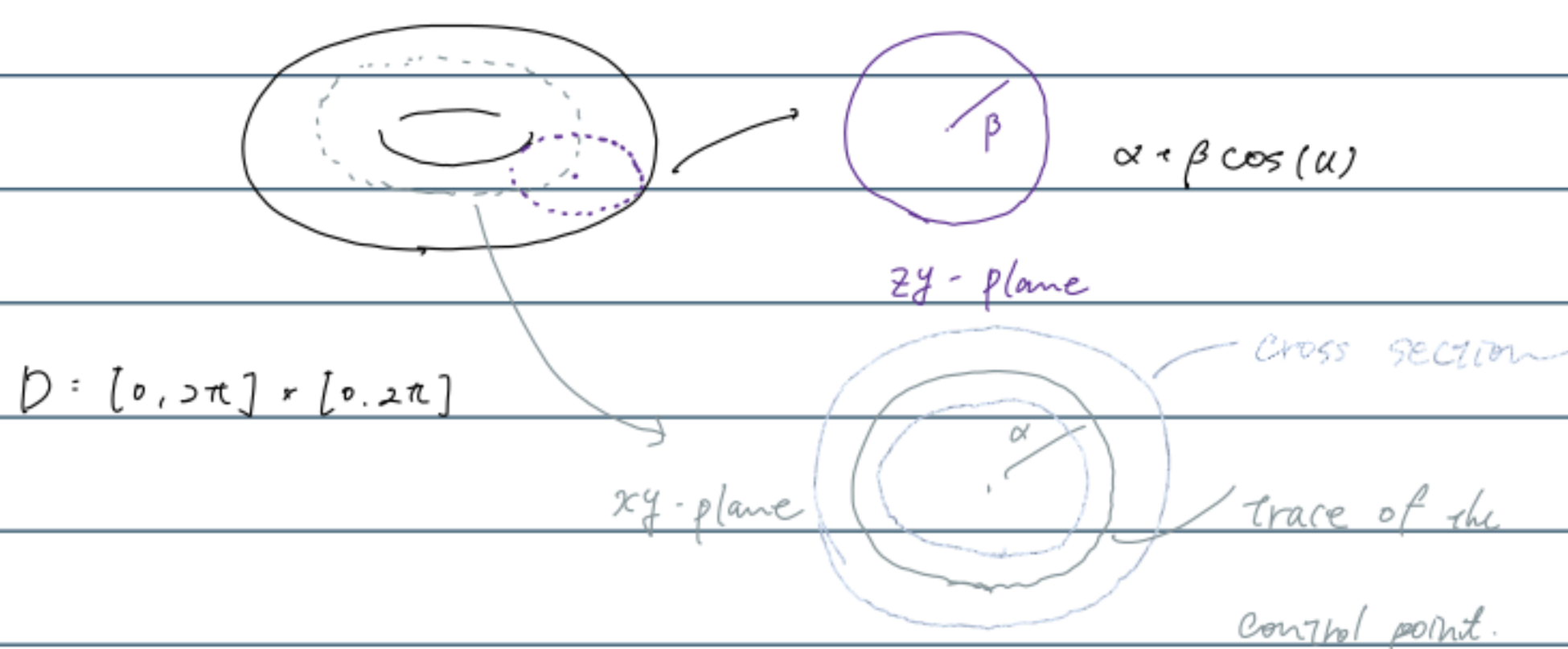
Last time: a surface in \mathbb{R}^3 has the form $\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

on some domain $D \subseteq \mathbb{R}^2$

Ex. The torus w/ major radius $\alpha \geq 0$ and minor radius β

(w/ $\alpha > \beta > 0$) is the surface

$$\vec{S}(u,v) = \langle (\alpha + \beta \cos(u)) \cos(v), (\alpha + \beta \cos(u)) \sin(v), \beta \sin(u) \rangle$$



1. Tangent planes

The tangent plane to surface $\vec{S}(u,v)$ at point $(a,b) \in D$

has normal vector $\vec{n}(a,b) = \vec{S}_u(a,b) \times \vec{S}_v(a,b)$

Note. $\vec{S}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$ can also be written $\frac{\partial \vec{S}}{\partial u}$.

Ex. Consider the torus $\vec{S}(u,v)$ w/ major radius 10 and minor radius 5.

What's the tangent plane to $S(u,v)$ at $(\frac{\pi}{4}, \frac{3}{4}\pi)$?

Ans. $\vec{S}(u,v) = \langle (10 + 5 \cos(u)) \cos(v), (10 + 5 \cos(u)) \sin(v), 5 \sin(u) \rangle$

$$\vec{S}_u = \langle -5 \sin(u) \cos(v), -5 \sin(u) \sin(v), 5 \cos(u) \rangle$$

$$\vec{S}_v = \langle -(10 + 5 \cos(u)) \sin(v), (10 + 5 \cos(u)) \cos(v), 0 \rangle$$

$$\vec{n}(u,v) = \vec{S}_u(u,v) \times \vec{S}_v(u,v)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \sin u \cos v & -5 \sin u \sin v & 5 \cos u \\ -(10 + 5 \cos u) \sin v & (10 + 5 \cos u) \cos v & 0 \end{vmatrix}$$

$$\begin{vmatrix} 5 \sin u \cos v & -5 \sin u \sin v & 5 \cos u \\ -(10 + 5 \cos u) \sin v & (10 + 5 \cos u) \cos v & 0 \end{vmatrix}$$

$$= -5 (10 + 5 \cos u) \langle \cos(u) \cos(v), \cos(u) \sin(v), \sin(u) \rangle$$

at every $(u, v) \in \text{dom}(\vec{S})$, this is the normal vector at $S(u, v)$

Now at the point

$$\begin{aligned} \vec{S}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) &= \langle (10 + 5 \cos(\frac{\pi}{4})) \cos(\frac{3\pi}{4}), (10 + 5 \cos(\frac{\pi}{4})) \sin \frac{3\pi}{4}, \sin \frac{\pi}{4} \rangle \\ &= \left\langle -\frac{10}{\sqrt{2}} - \frac{5}{2}, \frac{10}{\sqrt{2}} - \frac{5}{2}, \frac{1}{\sqrt{2}} \right\rangle \end{aligned}$$

We have normal vector

$$\begin{aligned} \vec{n}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) &= -5 \left(10 + \frac{5}{\sqrt{2}}\right) \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= -25 \left(2 + \frac{1}{\sqrt{2}}\right) \left\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle \end{aligned}$$

\therefore The tangent plane at this point is given by

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

$$\text{i.e. } \vec{n}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cdot (\vec{x} - \vec{S}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)) = 0$$

$$\text{i.e. } -25 \left(2 + \frac{1}{\sqrt{2}}\right) \cdot \left\langle x + \frac{10}{\sqrt{2}} + \frac{5}{2}, y - \frac{10}{\sqrt{2}} - \frac{5}{2}, z - \frac{1}{\sqrt{2}} \right\rangle \cdot \left\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\text{i.e. } -\frac{1}{2} \left(x + \frac{10}{\sqrt{2}} + \frac{5}{2}\right) + \frac{1}{2} \left(y - \frac{10}{\sqrt{2}} - \frac{5}{2}\right) + \frac{1}{\sqrt{2}} \left(z - \frac{1}{\sqrt{2}}\right) = 0.$$

II. Surface Area

The surface area of a surface $\vec{S}(u, v)$ parameterised on domain D is

$$A = \iint_D |\vec{S}_u \times \vec{S}_v| dA$$

Q: where is the formula coming from?

A: Piecewise approximation of Surface S via parallelograms.

limiting these approximations yields that formula

Note: for this to work, we assume that $\vec{S}(u, v)$

surface once on D .

Ex. Compute the surface area of the torus w/ major radius 10 & minor 5.

$$\begin{aligned} \text{Ans. } \vec{n}(u,v) &= \vec{S}_u(u,v) \times \vec{S}_v(u,v) = -5(10+5\cos(u)) \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u) \rangle \\ |\vec{S}_u(u,v) \times \vec{S}_v(u,v)| &= |-5(10+5\cos(u))| \sqrt{\cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u)} \\ &= 25|2+\cos(u)| \sqrt{\cos^2(u)(\cos^2(v) + \sin^2(v)) + \sin^2(u)} \\ &= 25(2+\cos(u)) \end{aligned}$$

$$\begin{aligned} \therefore \text{Area}(S) &= \iint_D |\vec{S}_u \times \vec{S}_v| dA \\ &= \int_{u=0}^{2\pi} \int_{v=0}^{2\pi} 25(2+\cos(u)) dv du \\ &= 206\pi^2 \end{aligned}$$

Note: If $f(x,y)$ is a function, the graph is a surface

$\vec{S}(x,y) = \langle x, y, f(x,y) \rangle$. The normal vector to this surface is

$$\vec{n}(x,y) = \vec{S}_x \times \vec{S}_y$$

$$= \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \langle -f_x, -f_y, 1 \rangle$$

$$\therefore \text{Area}(\text{graph } f) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA$$

Idea: Surface area is an area. So we should be able to write

$$\text{Area}(S) = \iint_S 1 ds$$

↳ resembles formula $\text{Area}(R) = \iint_R 1 dA$

to make this analogy work, $ds = \underbrace{|\vec{S}_u \times \vec{S}_v|}_{\text{Jacobian}} dA$

III. Surface Integrals

The integral of function $f(x,y,z)$ over surface S parameterised by $\vec{S}(u,v)$ on D is

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by $\vec{S}(u,v)$ on D is

$$\int_S f \, ds = \int_D f(S(u,v)) |\vec{S}_u \times \vec{S}_v| \, dA$$